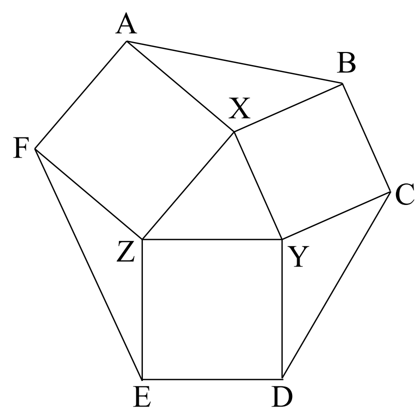
POMONA-WISCONSIN MATHEMATICS TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (2008-2009)

1. Let $p = 101010\cdots01$ be the m-digit number in which the first and last digits are 1 and the digits alternate between 1 and 0. For which positive integers m is the number p prime.

SOLUTION. First, the number m of digits must be odd since both the first and last digit of p is 1. Next, $11p = 10p + p$ is the number $111\cdots1$ with $m+1$ digits all equal to 1. Thus $11p = (10^{m+1} - 1)/9$, and we have $99p = 10^{m+1} - 1$. Since $m+1$ is even, we can write $m+1 = 2a$ with $a \geq 1$, and we obtain $99p = 10^{2a} - 1 = (10^a - 1)(10^a + 1)$. If p is prime, then p must divide one of the factors $10^a - 1$ or $10^a + 1$, and in either case, we get $p \leq 10^a + 1$. Since $99p = (10^a - 1)(10^a + 1)$, we deduce that $99 \geq 10^a - 1$. But $99 = 10^2 - 1$, so we see that $a \leq 2$. If $a = 2$, then $m = 3$ and $p = 101$ is indeed a prime number. The only other possibility is $a = 1$, in which case $m = 1$ and $p = 1$. Since the number 1 is not prime, by definition, the only solution is $m = 3$.

2. In the figure on the right, hexagon $ABCDEF$ is divided into three squares and four triangles. Show that the areas of all four triangles are equal.



SOLUTION. Let X, Y and Z be the third vertices of the triangles with side $\overline{AB}$, $\overline{CD}$ and $\overline{EF}$, respectively. We will prove that the area of $\triangle ABX$ is equal to the area of $\triangle XYZ$. Similar arguments show that the areas of $\triangle CDY$ and $\triangle EFZ$ also equal the area of $\triangle XYZ$, and thus all four triangles have equal areas.

Imagine rotating $\triangle XYZ$ about vertex X so as to make $\overline{XY}$ coincide with $\overline{XB}$. This is possible since $\overline{XY}$ and $\overline{XB}$ are two sides of a square, and hence have equal lengths. Also note that this is a rotation through $90°$ and that $\angle AXZ = 90°$. In particular, if Z' is the new position of point Z, then $\angle AXZ' = 180°$. In other words, points A, X and Z' lie on a line, and we see that $\overline{BX}$ is a median of $\triangle AZ'B$, since $AX = XZ = XZ'$.

Now, a median of a triangle always divides the triangle into two triangles with equal area, and thus the areas of $\triangle ABX$ and of $\triangle XBZ'$ are equal. The latter triangle, however, is just a rotation of $\triangle XYZ$ and so its area is equal to that of $\triangle XYZ$. It follows that $\triangle ABX$ and $\triangle XYZ$ have equal areas, as wanted.

3. Find all positive integers x and y such that $x \leq y \leq 2x$ and $1 + x^2 + y^2 = 3xy$.

SOLUTION. Since $x \leq y \leq 2x$, we see y is at most distance $x/2$ from $3x/2$, or equivalently that $|y - 3x/2| \leq x/2$. Squaring the latter yields $y^2 - 3xy + 9x^2/4 \leq x^2/4$, and thus $y^2 - 3xy + 2x^2 \leq 0$. But $y^2 - 3xy + 2x^2 = x^2 + (y^2 - 3xy + x^2) = x^2 - 1$, so the previous inequality yields $x^2 - 1 \leq 0$ and, since x is a positive integer, we conclude that $x = 1$. Furthermore, since $1 = x \leq y \leq 2x = 2$, we see that $y = 1$ or 2 are the only possibilities. Finally, we note that the pairs $x = 1$, $y = 1$ and $x = 1$, $y = 2$ both satisfy the equation $1 + x^2 + y^2 = 3xy$ and consequently we have found all solutions to the given equation and inequalities.

4. Your calculator will show that the number

$$\sqrt[3]{7+5\sqrt{2}}+\sqrt[3]{7-5\sqrt{2}}$$

is approximately an integer. Decide whether or not it is exactly an integer, and prove that your answer is correct.

SOLUTION. Let $a = \sqrt[3]{7+5\sqrt{2}}$ and $b = \sqrt[3]{7-5\sqrt{2}}$, and write $s = a + b$. Our task is to determine whether or not s is an integer. Observe that $a^3 + b^3 = 14$. Also, ab is the cube root of $(7+5\sqrt{2})(7-5\sqrt{2}) = 49 - 25{\cdot}2 = -1$, and thus $ab = -1$. Since $(a + b)^3 = a^3 + b^3 + 3ab(a + b) = 14 - 3(a + b)$, it follows that $s^3 + 3s - 14 = 0$.

Your calculator says that s is approximately 2, so we might guess that $s = 2$, and we check that 2 really is a solution of the cubic equation $s^3+3s-14 = 0$. This does not complete the proof, however, because we must consider the possibility that our cubic equation also has other roots near 2. Since $s = 2$ is a root, we see that $(s-2)$ must be a factor of $s^3 + 3s - 14$, and we calculate by long division that $s^3 + 3s - 14 = (s - 2)(s^2 + 2s + 7)$. Thus if s is not 2, then $s^2 + 2s + 7 = 0$. But this quadratic equation has no real root, so the only possibility is $s = 2$, which, of course, is an integer.

5. Let F_n be the nth Fibonacci number. Thus $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, and in general for $n > 2$, we have $F_n = F_{n-1} + F_{n-2}$. Now for each integer $n \geq 1$, write $A_n = (F_{n+1})^2 - (F_n)^2 - F_nF_{n+1}$ and $B_n = (F_n)^2 + (F_{n+2})^2 - 3F_nF_{n+2}$. Find simple formulas for A_n and B_n and use them to compute A_{1000} and B_{1000}.

SOLUTION. Let $n > 1$ and use the fact that $F_{n+1} = F_n + F_{n-1}$ to compute that

$$A_n = (F_{n+1})^2 - (F_n)^2 - F_nF_{n+1} = (F_{n-1} + F_n)^2 - (F_n)^2 - F_n(F_{n-1} + F_n)\,.$$

Simplifying this, we get $A_n = (F_{n-1})^2+F_{n-1}F_n-(F_n)^2$, and this is exactly $-A_{n-1}$. Since $A_1 = -1$, we deduce that $A_2 = 1$, $A_3 = -1$ and in general $A_n = (-1)^n$. Thus $A_{1000} = 1$.

Also

$$B_n = (F_n)^2 + (F_{n+2})^2 - 3F_nF_{n+2} = (F_n)^2 + (F_n + F_{n+1})^2 - 3F_n(F_n + F_{n+1})\,.$$

Simplifying this, we get $B_n = -(F_n)^2 - F_nF_{n+1} + (F_{n+1})^2$, and so $B_n = A_n$. In particular, $B_n = (-1)^n$ and $B_{1000} = 1$.

If n is odd and we write $x = F_n$ and $y = F_{n+2}$, then we have $-1 = B_n = x^2 + y^2 - 3xy$. Thus x and y satisfy the equation $1 + x^2 + y^2 = 3xy$ of Problem 3, and we conclude that this equation has infinitely many pairs of positive integer solutions.