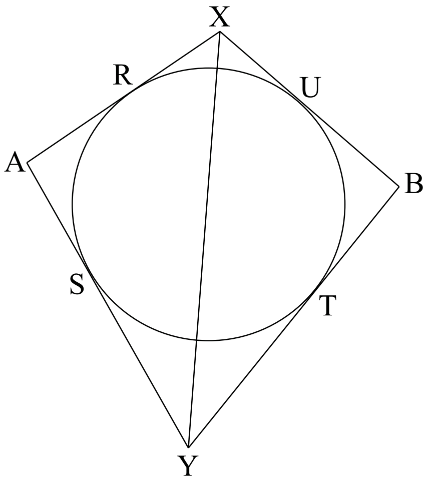
POMONA-WISCONSIN MATHEMATICS TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (2009-2010)

1. Given a positive integer N, write N^* to denote the integer obtained by adding N and all the digits of N. Thus $5^* = 10$ and $86^* = 100$. Also, $977^* = 1{,}000$ and $9968^* = 10{,}000$. Find an integer N such that $N^* = 1{,}000{,}000$, or prove that no such integer exists.

SOLUTION. Suppose that $N^* = 1{,}000{,}000$. Then N has at most six digits, so its digit sum is at most 54. Thus $N \geq 1{,}000{,}000 - 54 = 999{,}946$, and we can write $N = 999{,}9ab$, where a and b are digits. Then $1{,}000{,}000 = N^* = (999{,}900 + 10a + b) + (36 + a + b)$, and this yields $11a + 2b = 64$. Since $b \leq 9$, we have $11a \geq 64 - 18 = 46$, and since $b \geq 0$, we have $11a \leq 64$. Thus $46/11 \leq a \leq 64/11$, and since a is an integer, we must have $a = 5$. Then $2b = 64 - 55 = 9$, and this is a contradiction since b must be an integer. Therefore, we conclude that there is no integer N such that $N^* = 1{,}000{,}000$.

2. In the figure, lines $\overline{AX}$ and $\overline{AY}$ are perpendicular tangents to a circle. Also, $\overline{BX}$ and $\overline{BY}$ are perpendicular tangents to the same circle. Prove that line $\overline{XY}$ goes through the center of the circle.



SOLUTION. Let R, S, T and U be the four points of tangency, and let r, s, t and u be the lengths of the sides of the quadrilateral $XAYB$. Then $r = XR + AR$, $s = AS + YS$, $t = YT + BT$ and $u = BU + XU$. Since the lengths of the two tangents to a circle from the same point are equal, we have

$$\begin{aligned} r + t &= XR + AR + YT + BT \\ &= XU + AS + YS + BU = u + s\,. \end{aligned}$$

Furthermore, $\angle A = 90^\circ = \angle B$, so the Pythagorean theorem yields $r^2 + s^2 = (XY)^2 = t^2 + u^2$. Thus $r^2 - t^2 = u^2 - s^2$, and dividing through by $r + t = u + s$ yields $r - t = u - s$. Now adding this to the equation $r + t = u + s$, we obtain $2r = 2u$, so $r = u$, and hence $t = s$. Then $\triangle XAY \cong \triangle XBY$ by side-side-side, and thus line $\overline{XY}$ bisects $\angle AXB$. Since the bisector of the angle formed by two tangents of a circle from the same point goes through the center of the circle, it follows that $\overline{XY}$ goes through the center, as required.

3. Let a be a real number, and suppose that two of the three solutions of the cubic equation $x^3 + 3x^2 - 34x = a$ differ by 1. Find all possibilities for a.

SOLUTION. Suppose that both r and $r + 1$ are solutions to the equation $x^3 + 3x^2 - 34x = a$. Then $r^3 + 3r^2 - 34r = a$, and also $(r + 1)^3 + 3(r + 1)^2 - 34(r + 1) = a$. Subtracting the first of these equations from the second yields $(3r^2 + 3r + 1) + 3(2r + 1) - 34 = 0$, and simplification yields $3r^2 + 9r - 30 = 0$. Thus $0 = r^2 + 3r - 10 = (r + 5)(r - 2)$. We conclude that either $r = -5$ or

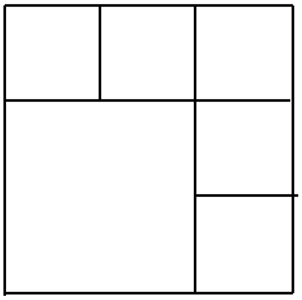
$r = 2$. If $r = -5$, then $a = (-5)^3 + 3(-5)^2 - 34(-5) = -125 + 75 + 170 = 120$, and we can check that $r + 1 = -4$ is also a solution to the equation $x^3 + 3x^2 - 34x = 120$. The other possibility is $r = 2$, which yields $a = 8 + 12 - 68 = -48$. Finally, we can check that $r + 1 = 3$ is also a solution to $x^3 + 3x^2 - 34x = -48$. Thus $a = 120$ and $a = -48$ are the two possibilities.

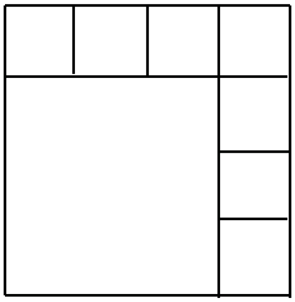
4. Given a positive integer n, find all polynomials $f(x)$ such that $f(0) = 1$ and $f(x)^2 + 4x^{n+1} = g(x)^2 + 4x^n$ for some polynomial $g(x)$.

SOLUTION. Plugging $x = 0$ into the equation $f(x)^2 + 4x^{n+1} = g(x)^2 + 4x^n$, and recalling that $f(0) = 1$, we obtain $g(0)^2 = 1$. Replacing g by $-g$ if necessary, we can assume that $g(0) = 1$. Now $(f + g)(f - g) = f^2 - g^2 = 4x^n - 4x^{n+1} = 4x^n(1 - x)$, and thus $f + g$ is a polynomial divisor of $4x^n(1 - x)$. But $f(x) + g(x)$ has the value 2 when $x = 0$, and thus x cannot be a factor of $f + g$. It follows that either $f + g = c$ or $f + g = c(1 - x)$ for some constant c. Plugging in $x = 0$ yields $2 = c$, and we conclude that either $f + g = 2$ or $f + g = 2(1 - x)$. If $f + g = 2$, then since $(f + g)(f - g) = 4x^n(1 - x)$, we deduce that $f - g = 2x^n(1 - x)$. Adding these two equations, we get $2f = 2 + 2x^n(1 - x)$, and thus $f(x) = 1 + x^n - x^{n+1}$. The remaining possibility is that $f + g = 2(1 - x)$, and thus $f - g = 2x^n$. Adding, as before, we get $2f = 2(1 - x) + 2x^n$, and this time we have $f(x) = 1 - x + x^n$.

5. Given a positive integer n other than 2, 3 or 5, show that a cardboard square can be cut into exactly n smaller squares, not necessarily of the same size.

SOLUTION. Say that a positive integer n is *good* if every square can be cut into n squares, not necessarily of the same size. Clearly 1 is good and the following diagrams show that both 6 and 8 are good.





To show that every positive integer other than 2, 3 or 5 is good, suppose this is false, and let n be the smallest integer other than 2, 3 or 5 that is not good. Since 1 is good, $n > 1$ and by assumption n is not 2 or 3, so $n > 3$. We next observe that $n - 3$ is not 2, 3 or 5. Indeed, if $n - 3 = 2$ then $n = 5$, contrary to our assumption. On the other hand, if $n - 3 = 3$ or 5, then $n = 6$ or 8, and we know both of these are good.

Finally, we show that n really is good after all. To this end, note that $n - 3$ is not 2, 3 or 5 and that it is smaller than n, so $n - 3$ must be good. Next, cut the original square into four equal smaller squares, and since $n - 3$ is good, cut one of those smaller squares into $n - 3$ squares. At this point, the original square has been cut into n squares, so n is good. We conclude from this contradiction that every positive integer different from 2, 3 or 5 is good.