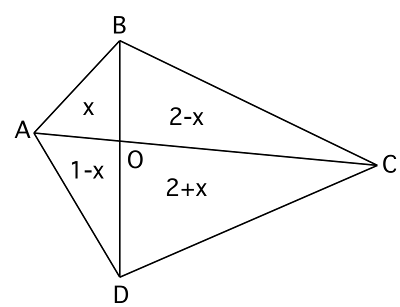
# POMONA-WISCONSIN MATHEMATICS TALENT SEARCH

## SOLUTIONS TO PROBLEM SET II (2007-2008)

**1.** Let $U$ be a set containing 29 objects and let $S_1, S_2, \ldots, S_{10}$ be 10 subsets of $U$, not necessarily distinct. Suppose that every 5 of the subsets taken together contain all 29 objects of $U$. Show that some three of these subsets taken together contain all 29 objects of $U$.

**SOLUTION.** Let us assume, by way of contradiction, that no three of the subsets taken together contain all 29 objects of $U$. Then for each triple $S_i, S_j, S_k$, with $i$, $j$ and $k$ distinct, there exists an object in $U$ that is not in $S_i$ or $S_j$ or $S_k$. We can therefore associate to each such triple an object of $U$ that avoids each subset. Now there are $\binom{10}{3} = (10{\cdot}9{\cdot}8)/(3{\cdot}2{\cdot}1) = 120$ triples and there are 29 objects in $U$, so the average number of triples that associate with the same object is $120/29 > 4$. In particular, there must be five triples which avoid the same object in $U$. Now these five triples cannot be formed out of four subsets $S_i, S_j, S_k$ and $S_\ell$, since the triples obtained from these four subsets come about by deleting one of the four. Thus these five triples must come from at least five of the subsets and, since these five subsets avoid the same object in $U$, they cannot possibly join together to yield of all $U$. This contradicts the given assumption on collections of five subsets and therefore there must be some three subsets that taken together contain all objects of $U$.

**2.** Let $ABCD$ be the convex quadrilateral, as shown, and let $O$ be the point of intersection of its two diagonals. Suppose the area of $\triangle ABD$ is 1, the area of $\triangle BCA$ is 2 and the area of $\triangle DAC$ is 3. Find the areas of $\triangle CDB$ and $\triangle ABO$.




**SOLUTION.** Let $x$ equal the area of $\triangle ABO$. Since this triangle and $\triangle BCO$ partition $\triangle BCA$ and since the latter triangle has area 2, it follows that the area of $\triangle BCO$ is equal to $2 - x$. Similarly, since $\triangle ABD$ has area 1, we see that $\triangle DAO$ has area equal to $1 - x$. Finally, since $\triangle DAC$ has area 3, we conclude that the area of $\triangle CDO$ is equal to $3 - (1 - x) = 2 + x$. In particular, the area of $\triangle CDB$ is $(2 - x) + (2 + x) = 4$.

Now $\triangle ABO$ and $\triangle BCO$ share the same altitude to $\overline{AC}$, so their areas are proportional to the lengths of their bases, namely $AO$ and $OC$. Similarly, the areas of $\triangle DAO$ and $\triangle CDO$ are also proportional to $AO$ and $OC$. Thus

$$\frac{x}{2-x} = \frac{AO}{OC} = \frac{1-x}{2+x}$$

and hence $2x + x^2 = x(2 + x) = (1 - x)(2 - x) = x^2 - 3x + 2$. We conclude that $5x = 2$ and therefore the area of $\triangle ABO$ is $x = 2/5$.

**3.** Find all positive integers $a$ and $b$ with $a! + 4! = b^2$.

**SOLUTION.** If $a \geq 6$, then $a! = 1{\cdot}2{\cdot}3{\cdot}4{\cdot}5{\cdot}6 \cdots = 4!{\cdot}(2k)$ for some even integer $2k$. Consequently, $a! + 4! = 4!{\cdot}(2k + 1) = 8{\cdot}3{\cdot}(2k + 1)$ and this number cannot be a perfect square since it is an odd multiple of $8 = 2^3$. It follows that $a \leq 5$, and for these numbers we have $1! + 4! = 25$, $2! + 4! = 26$, $3! + 4! = 30$, $4! + 4! = 48$ and $5! + 4! = 144$. Therefore, the only possibilities are $a = 1$, $b = 5$ and $a = 5$, $b = 12$.

**4.** For a fixed positive integer $n$, let $x_1, x_2, \ldots, x_n$ be $n$ positive real numbers that sum to 1. Find the smallest possible value for the sum of the reciprocals of these $n$ numbers.

**SOLUTION.** If $x_1 = x_2 = \cdots = x_n = 1/n$, then these numbers sum to 1, each reciprocal is equal to $n$, and hence the sum of the reciprocals is $n^2$. Thus $n^2$ is a possible value for the sum of the reciprocals, and we now show, by induction on $n$, that it is the smallest possible value. The case $n = 1$ is obvious.

Let $n \geq 2$ and let $x_1, x_2, \ldots, x_n$ be $n$ arbitrary positive real numbers that sum to 1. Then $x_1/(1-x_n), x_2/(1-x_n), \ldots, x_{n-1}/(1-x_n)$ are $n-1$ positive numbers that sum to 1, so by induction,

$$\frac{1-x_n}{x_1} + \frac{1-x_n}{x_2} + \cdots + \frac{1-x_n}{x_{n-1}} \geq (n-1)^2.$$

Setting $z = x_n$, we conclude that

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{n-1}} + \frac{1}{x_n} \geq \frac{(n-1)^2}{1-z} + \frac{1}{z}.$$

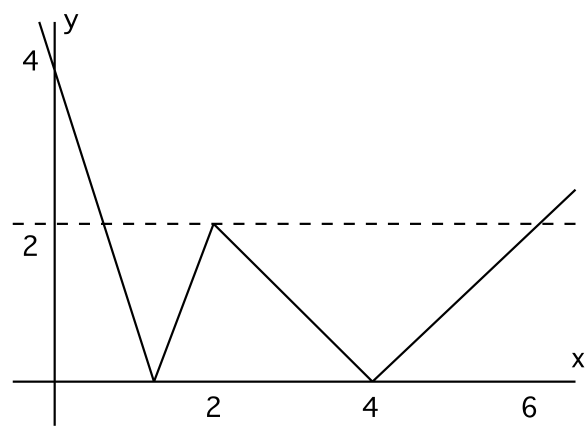
Thus it suffices to show that $(n-1)^2/(1-z) + 1/z \geq n^2$ when $0 < z < 1$, and by cross multiplying and simplifying, this is equivalent to $-2nz + 1 \geq -n^2z^2$. In turn, this is equivalent to $(nz-1)^2 = n^2z^2 - 2nz + 1 \geq 0$, so we are done.

**5.** Find all real numbers $a$ such that the equation

$$|\, x - |\, x - |x-4|\,|\,| = a$$

has exactly three real solutions $x$. Here, of course, $|x|$ is the absolute value of $x$.

**SOLUTION.** Let us draw the graph of $y = |\, x - |\, x - |x-4|\,|\,|$. Suppose first that $x \geq 4$. Then $|x-4| = x-4$, so $x - |x-4| = 4$ and this part of the graph lies along the line $y = |\, x - |\, x - |x-4|\,|\,| = |x-4| = x-4$. Next, let $2 \leq x \leq 4$. Then $|x-4| = 4-x$, so $x - |x-4| = x - (4-x) = 2x - 4 \geq 0$. Thus $|\, x - |x-4|\,| = 2x-4$, and this time we have the line $y = |\, x - |\, x - |x-4|\,|\,| = |\, x - (2x-4)\,| = |4-x| = 4-x$. Finally, suppose $x \leq 2$. Then $|x-4| = 4-x$, so $x - |x-4| = x - (4-x) = 2x-4 \leq 0$. Thus $|\, x - |x-4|\,| = 4-2x$, and $y = |\, x - |\, x - |x-4|\,|\,| = |\, x - (4-2x)\,| = |3x-4|$. Here the curve consists of two lines, namely $y = 3x-4$ when $4/3 \leq x \leq 2$, and $y = 4-3x$ when $x \leq 4/3$. A sketch of the curve is shown at the right.




Since the number of solutions in $x$ of the equation $|\, x - |\, x - |x-4|\,|\,| = a$ is precisely the number of points where the horizontal line $y = a$ crosses the curve $y = |\, x - |\, x - |x-4|\,|\,|$, we see from this drawing that $a = 2$ is the unique answer. Indeed, if $a > 2$ or $a = 0$ there are just two crossings, while if $0 < a < 2$, there are four crossing points. Of course, when $a < 0$, there are no crossings.