

POMONA-WISCONSIN MATHEMATICS TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (2008-2009)

1. A string of 0s and 1s is said to be *palindromic* if it reads the same backwards and forwards. Thus, for example, 101 and 1001 are palindromic, but 1101 is not. Now let S be an infinitely long string of 0s and 1s with no beginning and no end. Show that S contains a palindromic string consisting of four or more consecutive symbols. Decide whether or not S must contain a palindromic string consisting of five or more consecutive symbols.

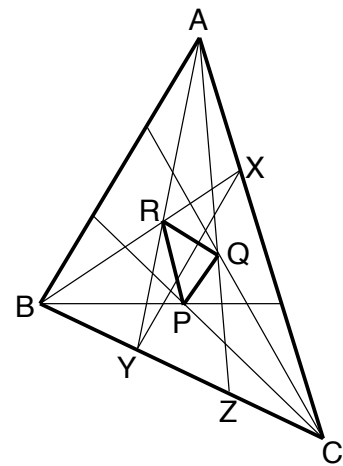
SOLUTION. If there are four consecutive 0s or 1s in S , we have a palindromic string, so we can assume that there are no four consecutive 0s or 1s in S . If S contains three consecutive 0s, they must therefore be preceded and followed by 1s, so we have the palindromic string 10001, and similarly, if S contains three consecutive 1s, it will contain the palindromic string 01110. We can assume, therefore, that S does not contain three consecutive 0s or 1s. If S contains two consecutive 0s, they must be preceded and followed by 1s, so S contains the palindromic string 1001, and similarly, if S contains two consecutive 1s, it contains the palindromic string 0110. We can thus assume that S does not contain two consecutive 0s or 1s, and thus the symbols 0 and 1 alternate in S , and S contains the palindromic string 01010. Thus in all cases, S contains a palindromic string of length at least four, indeed of length four or five.

There need not be a palindromic string of five or more consecutive symbols in S . To see this, suppose S is the string $\cdots 110010110010110010 \cdots$ formed by repeating the string 110010 infinitely often. Since the string 0011 does not occur in S , no palindromic string of consecutive symbols in S can contain 1100. All palindromic strings of consecutive symbols in S , therefore, must be part of 10010110. It is easy to check that no five or more consecutive symbols in 10010110 are palindromic.

2. In the diagram, each side of $\triangle ABC$ is trisected, and each vertex is joined by line segments to the two trisection points of the opposite side. Let P , Q and R be the intersections of three pairs of these line segments, as shown. Prove that $\triangle PQR$ is similar to $\triangle ABC$, and compute its area as a fraction of the area of the big triangle.

SOLUTION. We will show that $RQ = (1/5)BC$. It will then follow in the same way that $QP = (1/5)AB$ and $PR = (1/5)CA$, and thus $\triangle PQR$ is similar to $\triangle ABC$ by side-side-side, and its area is $(1/5)^2 = 1/25$ of the area of $\triangle ABC$.

Label points X , Y and Z , as shown, and draw \overline{XY} . Since $AX = (1/3)AC$ and $BY = (1/3)BC$, it follows that \overline{XY} is parallel to \overline{AB} and $XY = (2/3)AB$. We see now that $\triangle RXY$ and $\triangle RBA$ are similar, and each side of $\triangle RXY$ has length $2/3$ of the corresponding side of $\triangle RBA$. In particular, $RY = (2/3)RA$, and thus $AY = AR + RY = AR + (2/3)AR = (5/3)AR$. We now have $AR = (3/5)AY$, and similarly, $AQ = (3/5)AZ$. We conclude that $\triangle ARQ$ is similar to $\triangle AYZ$, and the length of each side of the smaller triangle is $3/5$ the corresponding side of the larger triangle. In particular, $RQ = (3/5)YZ = (3/5)(1/3)BC = (1/5)BC$, as claimed.



3. Find all positive integers n such that $(n^3 + 100)/(n^2 + 100)$ is an integer.

SOLUTION. Clearly, $n = 1$ is a solution. We show that there is no other solution by deriving a contradiction from the assumption that n is a solution and $n > 1$.

By assumption, $n^2 + 100$ is a divisor of $n^3 + 100$, and of course, it is also a divisor of $n(n^2 + 100) = n^3 + 100n$. It follows that $n^2 + 100$ divides $(n^3 + 100n) - (n^3 + 100) = 100(n - 1)$. Now $n^2 + 100$ exceeds 100, so it cannot divide 100, and since it divides $100(n - 1)$, it follows that some divisor $d > 1$ of $n^2 + 100$ must divide $n - 1$. Then d divides $(n + 1)(n - 1) = n^2 - 1$, and since d also divides $n^2 + 100$, it follows that d divides $(n^2 + 100) - (n^2 - 1) = 101$. But 101 is prime and d is a divisor exceeding 1, so we must have $d = 101$.

We now know that 101 divides $n - 1$ and therefore $n \geq 102$, since $n > 1$. On the other hand, since $n^2 + 100$ divides $100(n - 1)$ and $n > 1$, it follows that $n^2 + 100 \leq 100(n - 1)$. Thus $n^2 \leq 100(n - 2) < 100n$, so $n < 100$ and we have the required contradiction. We conclude that $n = 1$ is the only solution.

4. Let \square be an operation defined on the integers. (In other words, given integers a and b , we get an integer $a \square b$ determined by a and b .) Assume the following three axioms.

- (1) $(a + b)(a \square b) = (a^2) \square (b^2)$ for all integers a and b .
- (2) $(a \square b) + (b \square c) = a \square c$ for all integers a , b and c .
- (3) $1 \square 0 = 1$.

Show that $a \square b = a - b$ for all integers a and b .

SOLUTION. By (2), we have $(0 \square 0) + (0 \square 0) = 0 \square 0$, and thus $0 \square 0 = 0$. Using (2) again, we have $0 = 0 \square 0 = (0 \square a) + (a \square 0)$, and thus $0 \square a = -(a \square 0)$. Now let a and b be arbitrary integers. Then (2) yields $(a \square 0) + (0 \square b) = a \square b$, and if we multiply both sides by $a + b$, we obtain

$$a(a \square 0) + b(a \square 0) + a(0 \square b) + b(0 \square b) = (a + b)(a \square b) = (a^2) \square (b^2) \quad (*)$$

by axiom (1). Furthermore, by (1) again, $a(a \square 0) = (a + 0)(a \square 0) = (a^2) \square 0$, and $b(0 \square b) = (0 + b)(0 \square b) = 0 \square (b^2)$. The sum of the first and fourth terms on the left of (*) is therefore $((a^2) \square 0) + (0 \square (b^2)) = (a^2) \square (b^2)$, which is the same as the right side of (*). It follows that $b(a \square 0) + a(0 \square b) = 0$, and thus since $(b \square 0) = -(0 \square b)$, we have $b(a \square 0) = a(b \square 0)$ for all a and b .

Finally, letting $b = 1$, we have $a \square 0 = a(1 \square 0) = a$, using (3). This holds for all integers a , so we can write $a \square b = (a \square 0) + (0 \square b) = (a \square 0) - (b \square 0) = a - b$, as required.

5. Let n be a positive integer. Show that $K_n = 2^{2n-1} - 9n^2 + 21n - 14$ is a multiple of 27.

SOLUTION. Write $D_n = K_{n+1} - 4 \cdot K_n$ for $n \geq 1$. Suppose we can prove that all of the numbers D_n are multiples of 27. Since $K_{n+1} = 4 \cdot K_n + D_n$, it follows that if K_n is a multiple of 27, then so is K_{n+1} . Since $K_1 = 2 - 9 + 21 - 14 = 0$ is indeed a multiple of 27, it follows that K_2 is also, and thus so is K_3 a multiple of 27. Continuing in this way, it follows that all K_n are multiples of 27.

It remains to understand D_n . For this, observe that

$$\begin{aligned} D_n &= K_{n+1} - 4 \cdot K_n \\ &= (2^{2n+1} - 4 \cdot 2^{2n-1}) - 9((n+1)^2 - 4 \cdot n^2) + 21((n+1) - 4 \cdot n) - 14(1 - 4 \cdot 1) \\ &= 0 - 9(-3n^2 + 2n + 1) + 21(1 - 3n) - 14(-3) \\ &= 27(n^2 - 3n + 2), \end{aligned}$$

and therefore D_n is indeed a multiple of 27, as wanted.