POMONA-WISCONSIN MATHEMATICS TALENT SEARCH
SOLUTIONS TO PROBLEM SET IV (2007-2008)

1. If p is a prime, find all positive integer solutions p, a, b, c to the equation

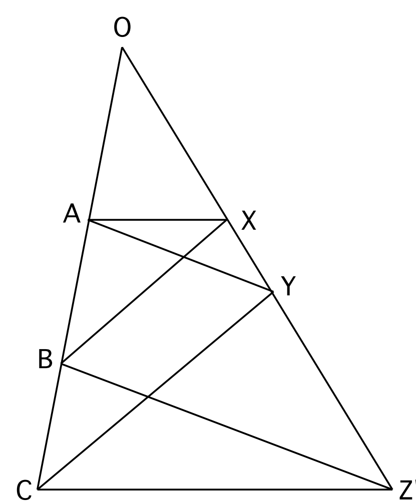
$$\frac{1}{p} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\,.$$

SOLUTION. Let p, a, b, and c satisfy the above displayed equation. Then multiplying that equation by $pa^2b^2c^2$ yields $(*)$ $p(a^2b^2 + b^2c^2 + c^2a^2) = a^2b^2c^2$. In particular, p divides $a^2b^2c^2$ and hence, since p is prime, p must divide at least one of these factors. By symmetry, assume that p divides a and write $a = p\bar{a}$ for some integer $\bar{a}$. Substituting $p\bar{a}$ for a in equation $(*)$ and dividing by p yields $p^2\bar{a}^2b^2 + b^2c^2 + p^2c^2\bar{a}^2 = p\bar{a}^2b^2c^2$, so p divides b^2c^2. Thus p also divides one of b or c. Again, by symmetry, we can assume that p divides b and we write $b = p\bar{b}$.

Now, from the original displayed equation, we see that $1/p > 1/c^2$, so $c^2 \geq p+1$ and $1/p - 1/c^2 \geq 1/p - 1/(p+1) = 1/p(p+1) > 1/(2p^2)$. Thus $1/(p^2\bar{a}^2) + 1/(p^2\bar{b}^2) = 1/p - 1/c^2 > 1/(2p^2)$ and hence $1/\bar{a}^2 + 1/\bar{b}^2 > 1/2$. Certainly, we cannot have both $\bar{a}$ and $\bar{b}$ strictly larger than 1, so say $\bar{b} = 1$ and $b = p\bar{b} = p$. Substituting into equation $(*)$ and dividing by p^3 then yields $(**)$ $p^2\bar{a}^2 + c^2 + c^2\bar{a}^2 = p\bar{a}^2c^2$, so $\bar{a}^2$ divides c^2 and hence $\bar{a}$ divides c. Similarly, c^2 divides $p^2\bar{a}^2$, so c divides $p\bar{a}$. It follows that $c = u\bar{a}$ where $u = 1$ or p.

Finally, substituting $c = u\bar{a}$ into equation $(**)$ and dividing by $\bar{a}^2$ yields $(***)$ $p^2 + u^2 + c^2 = pc^2$. If $u = 1$, then $(***)$ yields $(p-1)c^2 = p^2 + 1 = (p-1)(p+1) + 2$. Thus $(p-1)$ divides 2, so $p = 2$ or 3 and, in either case, we obtain $c^2 = 5$. This is a contradiction since c is an integer. We conclude that $u = p$, so $(***)$ with $c = u\bar{a} = p\bar{a}$ yields $(p-1)\bar{a}^2 = 2$. Thus again, $p = 2$ or 3, and since $p = 2$ implies that $\bar{a}^2 = 2$, a contradiction, we must have $p = 3$, $\bar{a} = 1$, $a = p\bar{a} = p = 3$, $c = p\bar{a} = p = 3$, and $b = p = 3$. In other words, $p = a = b = c = 3$ is the only possible solution and, as is easily checked, it is indeed a solution to the given equation.

2. Let A, B, C, X, Y, and Z be six distinct points in the plane with A, B, and C collinear. If $\overline{AX}$ is parallel to $\overline{CZ}$, $\overline{AY}$ is parallel to $\overline{BZ}$, and $\overline{BX}$ is parallel to $\overline{CY}$, prove that X, Y, and Z are collinear.



SOLUTION. If lines $\overline{XY}$ and $\overline{YZ}$ are both parallel to line $\overline{AC}$, then surely X, Y, and Z are collinear. Thus we can assume that one of these lines is not parallel to $\overline{AC}$ and, since the argument is the same in both cases, we can assume that it is $\overline{XY}$. This allows us to extend both lines $\overline{AC}$ and $\overline{XY}$ so that they meet at point O, as indicated. Furthermore, we let Z' be the point on line $\overline{XY}$, extended if necessary, so that $\overline{BZ'}$ is parallel to $\overline{AY}$.

Now $\overline{BX}$ is parallel to $\overline{CY}$, so $\angle OBX = \angle OCY$ and $\angle BXO = \angle CYO$. It follows by angle-angle-angle that $\triangle OBX$ is similar to $\triangle OCY$, and hence $OX/OY = OB/OC$. In the same way, since $\overline{AY}$ is parallel to $\overline{BZ'}$, we conclude that $\triangle OAY$ is similar to $\triangle OBZ'$, and hence $OY/OZ' = OA/OB$. Multiplying these two equalities of ratios yields $OX/OZ' = OA/OC$, and hence $\triangle OAX$ is similar to $\triangle OCZ'$ by the side-angle-side condition. In particular, $\angle OAX = \angle OCZ'$, so $\overline{AX}$ is parallel to $\overline{CZ'}$. Since $\overline{BZ}$ is parallel to $\overline{AY}$, it is clear that $\overline{BZ} = \overline{BZ'}$. Moreover, since $\overline{CZ}$ is parallel to $\overline{AX}$, it follows that $\overline{CZ} = \overline{CZ'}$. Thus, this pair of lines meets at both Z and Z', so we conclude that $Z = Z'$, and hence that X, Y, and Z are collinear.

3. (New Year's Problem) Let S be the set of all positive integers s such that $2^{2008} + 2^s + 1$ is a square. Find the smallest member of S and prove that it is not the only member of S.

SOLUTION. For any positive integer m, note that $(2^m+1)^2 = 2^{2m}+2^{m+1}+1$, and this expression can equal $2^{2008} + 2^s + 1$ by choosing $2m = 2008$ or $m+1 = 2008$. In the former case, $m = 1004$, so $s = m+1 = 1005$ is contained in S. In the latter case, $m = 2007$, so $s = 2m = 4014$ is in S. Thus S contains at least two members, namely 1005 and 4014. Now if $2^{2008} + 2^s + 1$ is a perfect square, then certainly its square root is larger than 2^{1004}. In other words, $2^{2008} + 2^s + 1 \geq (2^{1004} + 1)^2 = 2^{2008} + 2^{1005} + 1$. It follows that $s \geq 1005$, and therefore 1005 is the smallest member of S.

4. Find the functions $f(x)$ that satisfy $f(x) + f(1/(1-x)) = 1/x(1-x)$ for all real numbers $x \neq 0,\ 1$.

SOLUTION. We assume throughout that $x \neq 0$ or 1 and that $f(x)$ satisfies the above identity. Since $1/(1-x)$ is also not 0 or 1, we can replace x by $1/(1-x)$ in that formula. Then, $1/(1-x)$ and $1/x(1-x)$ become $(x-1)/x$ and $-(1-x)^2/x$, respectively, so we obtain a second identity, namely $f(1/(1-x)) + f((x-1)/x) = -(1-x)^2/x$. Furthermore, since $(x-1)/x$ is not 0 or 1, we can also replace x by $(x-1)/x$ in the original formula. Then $1/(1-x)$ becomes x, and $1/x(1-x)$ becomes $x^2/(x-1)$, so we obtain a third identity, namely $f((x-1)/x) + f(x) = x^2/(x-1)$. Finally, by adding the first and third identities and subtracting the second, we obtain $2f(x) = x^2/(x-1) + 1/x(1-x) + (1-x)^2/x = (2x^2 - x + 2)/x$, so $f(x) = x + 1/x - 1/2$. In other words, the latter is the only possibility for $f(x)$ and since, as is easily checked, it does satisfy the given identity, we conclude that $f(x) = x + 1/x - 1/2$ for $x \neq 0$ or 1.

5. Let n be a positive integer, and recall that the binomial coefficient $b(n) = \binom{2n}{n} = (2n)!/(n!\,n!)$ is also an integer. Show that $b(n)$ is always even and that it is divisible by 4 unless n is a power of 2.

SOLUTION. Let us write $e(n)$ for the exponent of 2 in the prime factorization of $n!$. In other words, $2^{e(n)}$ divides $n!$, but $2^{e(n)+1}$ does not. Note that $e(0) = 0$ since $0! = 1$, by definition. Now let $n \geq 1$. Then each even integer $\leq n$ contributes a factor of 2 to $n!$, and there are $\lfloor n/2 \rfloor$ such integers. Here, of course, $\lfloor x \rfloor$ is the largest integer less than or equal to the real number x. But some of these even integers $\leq n$ are divisible by 4, indeed there are $\lfloor n/4 \rfloor$ such, and each contributes an additional factor of 2 to $n!$. Similarly, there are $\lfloor n/8 \rfloor$ integers $\leq n$ that are divisible by 8, and each such also contributes an additional factor of 2 to n. Continuing in this manner, we conclude that $e(n) = \lfloor n/2 \rfloor + \lfloor n/4 \rfloor + \lfloor n/8 \rfloor + \cdots + \lfloor n/2^m \rfloor$, where m is the largest integer with $2^m \leq n$. We can stop at 2^m since all further terms are equal to zero. Since $\lfloor x \rfloor \leq x$ for all real numbers x, it follows that $e(n) \leq n/2 + n/4 + n/8 + \cdots + n/2^m = n - n/2^m$.

Now by plugging in $2n$ in place of n, we get $e(2n) = n + \lfloor n/2 \rfloor + \lfloor n/4 \rfloor + \lfloor n/8 \rfloor + \cdots + \lfloor n/2^m \rfloor = n + e(n)$. Thus, if $f(n)$ denotes the exponent of 2 in the prime factorization of the binomial coefficient $b(n)$, then $f(n) = e(2n) - 2e(n) = n - e(n) \geq n - (n - n/2^m) = n/2^m$. In particular, since $2^m \leq n$, we see that $f(n) \geq 1$ and thus $b(n)$ is even. Also, if n is not a power of 2, then $n > 2^m$ and hence $f(n) > 1$. But $f(n)$ is an integer, so $f(n) \geq 2$ and $b(n)$ is divisible by 4 in this case. On the other hand, if $n = 2^m$, then $e(n) = n/2 + n/4 + n/8 + \cdots + n/2^m = n - n/2^m$, so we conclude that $f(n) = n - e(n) = n/2^m = 1$ and consequently $b(n) = b(2^m)$ is not divisible by 4.