POMONA-WISCONSIN MATHEMATICS TALENT SEARCH

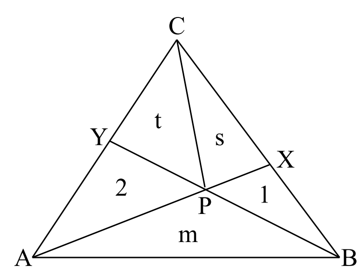
SOLUTIONS TO PROBLEM SET IV (2008-2009)

1. (**New Year's Problem**.) Find all positive integers n such that $n^2 + n + 2009$ is a square.

SOLUTION. Assume that $n^2+n+2009$ is a square. It is thus the square of an integer exceeding n, and we can write $n^2 + n + 2009 = (n + d)^2$, where $d \geq 1$. Simplification of this equation yields $n(2d - 1) = 2009 - d^2$, so $d^2 < 2009$, and we certainly have $d < 50$. Now $2d - 1$ is a divisor of $2009 - d^2$, so it is a divisor of $4(2009 - d^2) = 8036 - 4d^2$. But $2d - 1$ is also a divisor of $(2d-1)(2d+1) = 4d^2-1$, and thus $2d-1$ is a divisor of $(8036-4d^2)+(4d^2-1) = 8035 = (5)(1607)$. Now it is easy to check that 1607 is a prime number, so the only positive divisors of 8035 are 1, 5, 1607 and 8035, and thus $2d - 1$ must be one of these four numbers. But $d < 50$, so $2d - 1 < 100$, and the only possibilities are $2d - 1 = 1$ and $2d - 1 = 5$. Thus $d = 1$ or $d = 3$, and since $n = (2009 - d^2)/(2d - 1)$, we see that if $d = 1$ then $n = 2008$, and if $d = 3$, then $n = 2000/5 = 400$. Thus n must be one of the numbers 2008 or 400, and in either of these cases, it is easy to check that $n^2 + n + 2009$ is a square.

2. In the figure, the area of $\triangle ABC$ is a whole number. Lines $\overline{AX}$ and $\overline{BY}$ are drawn, where X lies on side $\overline{BC}$ and Y lies on side $\overline{AC}$, and these lines meet at point P, inside the triangle. The area of $\triangle BPX$ is 1, the area of $\triangle APY$ is 2, and the area of $\triangle APB$ is a whole number. Find the area of $\triangle ABC$, and prove that your answer is correct.

SOLUTION. Draw $\overline{PC}$ and let the areas of $\triangle ABP$, $\triangle PXC$ and $\triangle PYC$ be m, s and t, respectively, as indicated in the diagram. By hypothesis, m is an integer, and hence $s + t$ is an integer, which we call n. The ratio CX/BX is equal to the ratio of the area of $\triangle ACX$ to the area of $\triangle ABX$, and it is also equal to the ratio of the area of $\triangle PCX$ to the area of $\triangle PBX$. Thus $s/1 = (s+t+2)/(m+1)$, and we have $s = (n + 2)/(m + 1)$. Similarly, $t/2 = (s + t + 1)/(m + 2)$, and thus $t = 2(n + 1)/(m + 2)$. We therefore have



$$n = s + t = \frac{n+2}{m+1} + \frac{2n+2}{m+2}\,,$$

and solving for n, we get $n = (4m+6)/(m^2-2)$. Since $n \geq 1$, we conclude that $m^2 - 2 \leq 4m + 6$, and thus $m^2 - 4m \leq 8$. Then $(m - 2)^2 = m^2 - 4m + 4 \leq 12$, and since m is an integer, it follows that $m - 2 \leq 3$, so $m \leq 5$. As m runs over the set $\{1, 2, 3, 4, 5\}$, the corresponding values for $n = (4m + 6)/(m^2 - 2)$ are -10, 7, 18/7, 11/7 and 26/23. Since n is a positive integer, the only possibility is $m = 2$ and $n = 7$, and thus the area of $\triangle ABC$ is $1 + 2 + 2 + 7 = 12$.

3. Find a simple expression (in terms of n) for the sum S_n of all of the numbers of the form $k2^k$ where k is an integer and $1 \leq k \leq n$.

SOLUTION. A bit of experimentation shows that $S_1 = 2$, $S_2 = 10$, $S_3 = 34$, $S_4 = 98$, and $S_5 = 258$. If we subtract 2 from each of these sums, we get the numbers 0, 8, 32, 96 and 256, and we observe that these numbers tend to be divisible by large powers of 2. In fact, if we divide these numbers by 4, 8, 16, 32 and 64, respectively, the quotients are 0, 1, 2, 3 and 4. This suggests that if we divide $S_n - 2$ by 2^{n+1}, we get $n - 1$, and we have the formula $S_n = 2 + (n - 1)2^{n+1}$.

We prove that $S_n = 2 + (n-1)2^{n+1}$ by induction on n. We have already established this formula for $n \leq 5$, and we show now that if the formula holds for some integer n, it also holds for the next integer, $n+1$. Indeed, we have

$$\begin{aligned} S_{n+1} &= S_n + (n+1)2^{n+1} = 2 + (n-1)2^{n+1} + (n+1)2^{n+1} \\ &= 2 + \big((n-1) + (n+1)\big)2^{n+1} = 2 + (2n)2^{n+1} = 2 + n2^{n+2}\,, \end{aligned}$$

as wanted.

4. Decide (with proof) whether or not there exists a set $\mathcal{E}$ of even positive integers such that every even positive integer can be written in a unique way as a sum of distinct members of $\mathcal{E}$. Similarly, decide if there exists a set $\mathcal{O}$ of odd positive integers such that every odd positive integer can be written in a unique way as a sum of distinct members of $\mathcal{O}$.

SOLUTION. The set $\mathcal{E} = \{2, 4, 8, 16, \ldots\}$ has the desired property, where the members of $\mathcal{E}$ are all of the numbers 2^i, with $i \geq 1$. To see this, recall that every positive integer n has a unique binary expansion, and thus there is exactly one way to write n as a sum of distinct powers of 2, where we use 2^i as one of the summands precisely when the ith digit from the right is 1, and where we start counting from $i = 0$. The even integers are exactly those whose rightmost binary digit is 0, which means that they are uniquely sums of 2^i with $i > 0$.

There is no set $\mathcal{O}$ with the stated property. To see this, assume that $\mathcal{O}$ exists. Since 1, 3, 5 and 7 cannot be written as sums of distinct smaller odd positive integers, each of these four numbers must be a member of $\mathcal{O}$. Of course, 1, 3, 5 and 7 cannot exhaust $\mathcal{O}$ since there are odd integers that exceed $1+3+5+7$. We can thus choose $m \in \mathcal{O}$ with $m > 7$. The odd number $m+1+7 = m+3+5$ can thus be written in two different ways as a sum of distinct members of $\mathcal{O}$, and this contradicts the assumed uniqueness. We conclude that no such set $\mathcal{O}$ can exist.

5. Let n be a positive integer. A deck of $2n$ numbered cards contains exactly two cards marked with each of the integers from 1 to n, and these are arranged in the order $1, 1, 2, 2, 3, 3, \ldots, n, n$ from top to bottom. Observe that if $n = 3$, the deck can be cut into two pieces, namely $1, 1, 2, 2$ and $3, 3$, so that the sums of the numbers on the cards in the top and bottom parts are equal. Prove that there are infinitely many positive integers n for which the deck *cannot* be cut into two pieces so that the sums of the cards in the top and bottom parts are equal.

SOLUTION. The sum of the integers from 1 to n is $n(n+1)/2$, so the sum of all the numbers in our deck is twice this, namely $n(n+1)$. One way to cut the deck would be for the top part to contain cards numbered $1, 1, 2, 2, \ldots, m, m$ for some integer m, with the top card of the bottom part labeled $m+1$. Alternatively, the top part might contain $1, 1, 2, 2, \cdots, m$, for some integer m, where the top card of the bottom part is labeled m. In the first case, the sum of the cards in the top part is $m(m+1)$ and in the second, the sum is $m(m+1) - m = m^2$. If the sum of the cards in the top part of the deck is exactly half of the total of all of the cards, we either have $m(m+1) = n(n+1)/2$ or $m^2 = n(n+1)/2$. Our goal is to show that there are infinitely many positive integers n such that neither of these equations has an integer solution for m.

Suppose $n = 8k+5$ for some integer k. Then $n(n+1)/2 = (8k+5)(4k+3)$, and it is easy to see that this number leaves a remainder of 3 upon division by 4. In particular $n(n+1)/2$ is odd, so it cannot be of the form $m(m+1)$ for any integer m. Also, $n(n+1)/2$ cannot be a square since odd squares leave a remainder of 1 upon division by 4. It follows that $n(n+1)/2$ cannot equal either $m(m+1)$ or m^2, and so if n is any one of the infinitely many numbers of the form $8k+5$, then the deck cannot be cut into two pieces with equal sums.