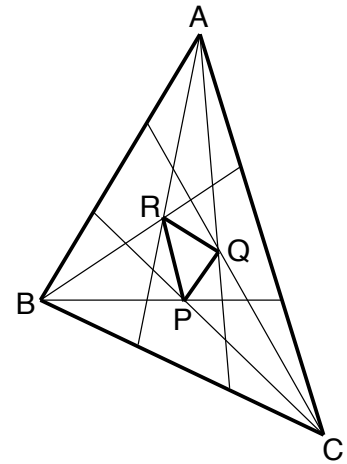


POMONA-WISCONSIN MATHEMATICS TALENT SEARCH

PROBLEM SET III (2008-2009)

DECEMBER 2008

1. A string of 0s and 1s is said to be *palindromic* if it reads the same backwards and forwards. Thus, for example, 101 and 1001 are palindromic, but 1101 is not. Now let  $S$  be an infinitely long string of 0s and 1s with no beginning and no end. Show that  $S$  contains a palindromic string consisting of four or more consecutive symbols. Decide whether or not  $S$  must contain a palindromic string consisting of five or more consecutive symbols.
2. In the diagram, each side of  $\triangle ABC$  is trisected, and each vertex is joined by line segments to the two trisection points of the opposite side. Let  $P$ ,  $Q$  and  $R$  be the intersections of three pairs of these line segments, as shown. Prove that  $\triangle PQR$  is similar to  $\triangle ABC$ , and compute its area as a fraction of the area of the big triangle.
3. Find all positive integers  $n$  such that  $(n^3 + 100)/(n^2 + 100)$  is an integer.
4. Let  $\square$  be an operation defined on the integers. (In other words, given integers  $a$  and  $b$ , we get an integer  $a \square b$  determined by  $a$  and  $b$ .) Assume the following three axioms.
  - (1)  $(a + b)(a \square b) = (a^2) \square (b^2)$  for all integers  $a$  and  $b$ .
  - (2)  $(a \square b) + (b \square c) = a \square c$  for all integers  $a, b$  and  $c$ .
  - (3)  $1 \square 0 = 1$ .



Show that  $a \square b = a - b$  for all integers  $a$  and  $b$ .

5. Let  $n$  be a positive integer. Show that  $K_n = 2^{2n-1} - 9n^2 + 21n - 14$  is a multiple of 27.

You are invited to submit a solution even if you get just one problem. Please do not write your solutions on this problem page. Remember that solutions usually require a proof or justification.

|                       |  |                                       |              |
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| <b>Return To</b>      | <b>MATHEMATICS TALENT SEARCH</b><br>Dept. of Mathematics, Pomona College<br>610 N. College Ave., Claremont, CA 91711-6348<br>mathsearch@pomona.edu<br>emailed solutions require the information below too! | <b>Deadline</b><br>January 5,<br>2009 |              |
| <b>Or Email To</b>    |  | <b>Problem</b>                        | <b>Score</b> |
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